Incorporating intuitive prior in image reconstruction

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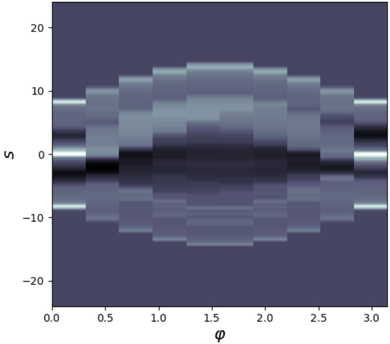
Joint work with Stanley Durrleman (ICM, France), Alain Trouvé (ENS Paris-Saclay, France), Ozan Öktem (KTH, Sweden)

INTRODUCTION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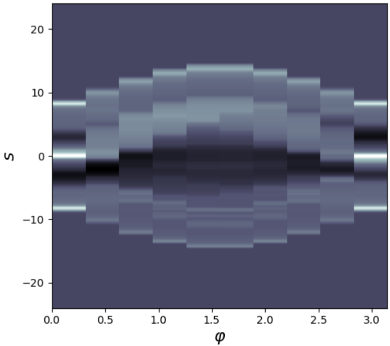
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with $T : X \mapsto Y$ (known forward operator)

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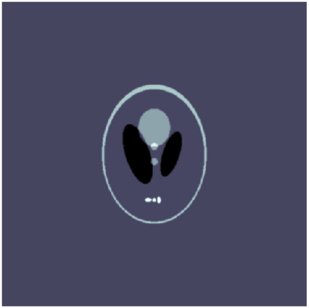

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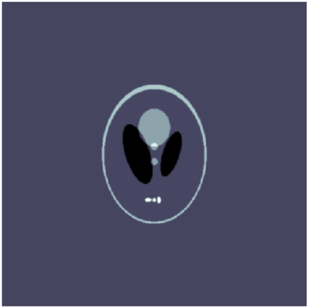

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$$\varphi \mapsto D\Big(T(\varphi \cdot I_0), g\Big) + R(\varphi)$$

Definition (Large deformation)

For $v \in L^1([0,1],V)$, we set φ^v **the flow of v**:

$$\begin{cases} \dot{\varphi}^v(t) & = & v(t) \circ \varphi^v(t) \\ \varphi^v(0) & = & Id \end{cases}$$

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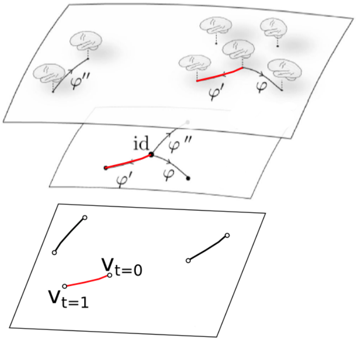
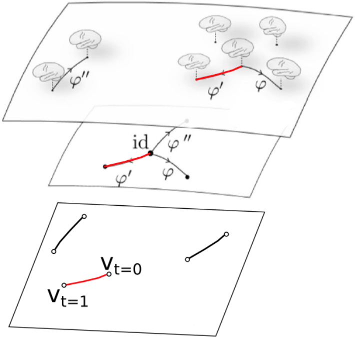
$\longrightarrow$ Action on images: $\varphi \cdot I = I \circ \varphi^{-1}$.



Diffeomorphometry and geodesic positioning systems for human anatomy, Miller et al, Technology 2014.



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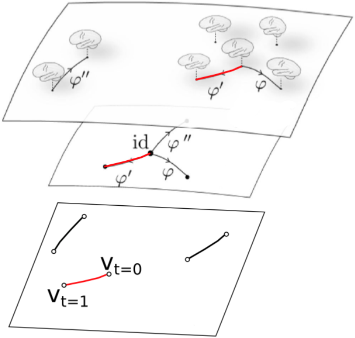
$$v \in L^2([0,1], V) \mapsto D\Big(T(\varphi^v_{t_1} \cdot I_0), g\Big) + \int_0^1 c(v_t)\mathrm{d}t$$



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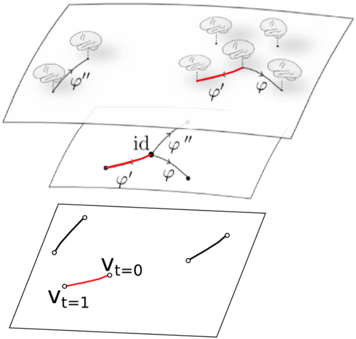

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[Sylvain Arguillere. Géométrie sous-riemannienne en dimension infinie et applications à l'analyse mathématique des formes. PhD thesis, Paris 6, 2014.]
[Alain Trouvé. Diffeomorphisms groups and pattern matching in image analysis. International Journal of Computer Vision, 28(3):213–221, 1998.]

Incorporating a structure in large deformations:

- Sparse LDDMM (Deformetrica) [S. Durrleman, M. Prastawa, G. Gerig, and S. Joshi. Optimal data-driven sparse parameterization of diffeomorphisms for population analysis. In Information Processing in Medical Imaging , pages 123-134. Springer, 2011]
- Higher order momentum [S. Sommer M. Nielsen, F. Lauze, and X. Pennec. Higher-order momentum distributions and locally affine lddmm registration. SIAM Journal on Imaging Sciences, 2013]
- GRID [U. Grenander , A. Srivastava , S. Saini. A pattern-theoric characerization of biological growth. IEEE, 2007]
- Poly-affine [V. Arsigny, X. Pennec, N. Ayache, 2005. Polyrigid and Polyaffine Transformations: A Novel Geometrical Tool to Deal with Non-rigid Deformations – Application to the Registration of Histological Slices. Medical Image Analysis 9, 507–523]
- Diffeons [L. Younes. Constrained diffeomorphic shape evolution. Foundations of Computational Mathematics, 2012.]

1. Defining easily complex generators
2. Evolution of generators during integration of the flow
3. Ensuring mathematical properties

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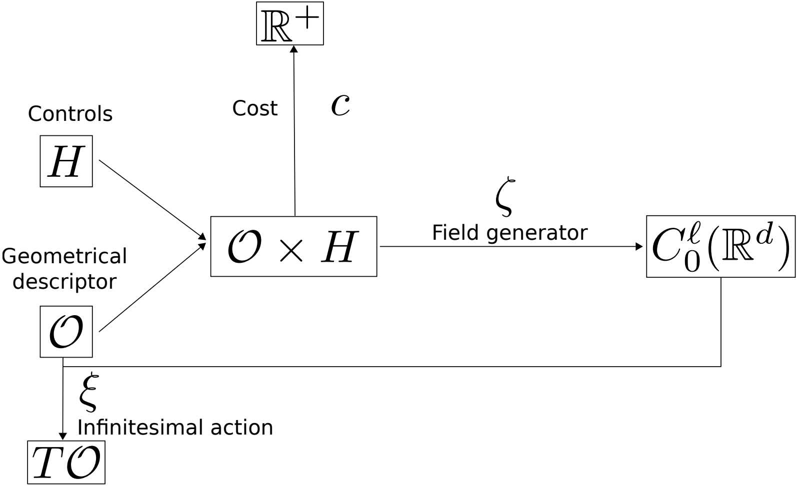
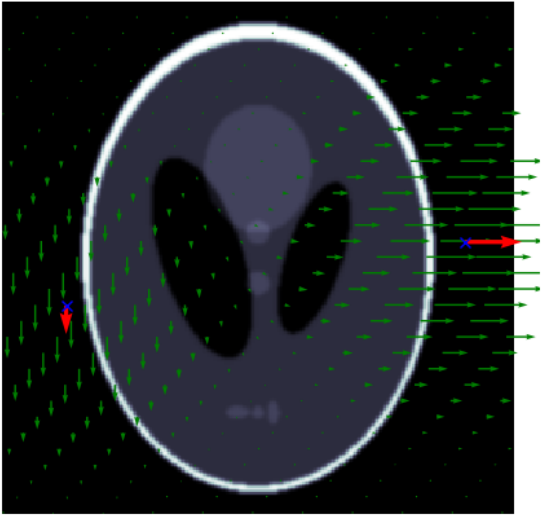
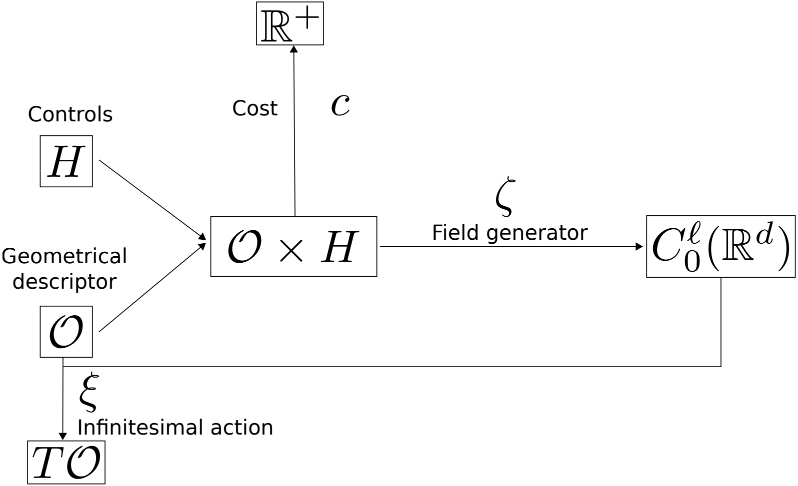
STRUCTURED VECTOR FIELDS
USING A DEFORMATION PRIOR

$$M = (\mathcal{O}, H, \zeta, \xi, c)$$



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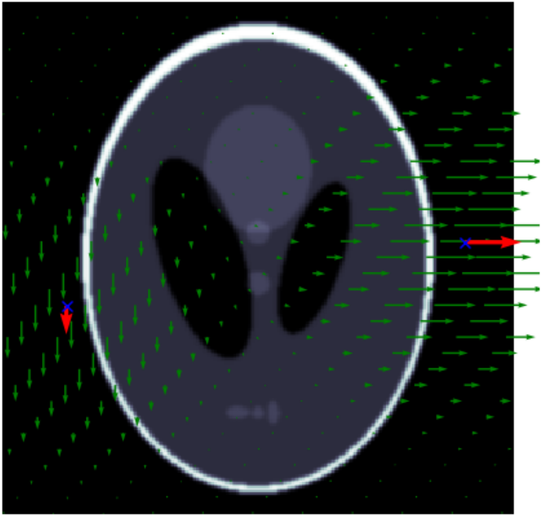

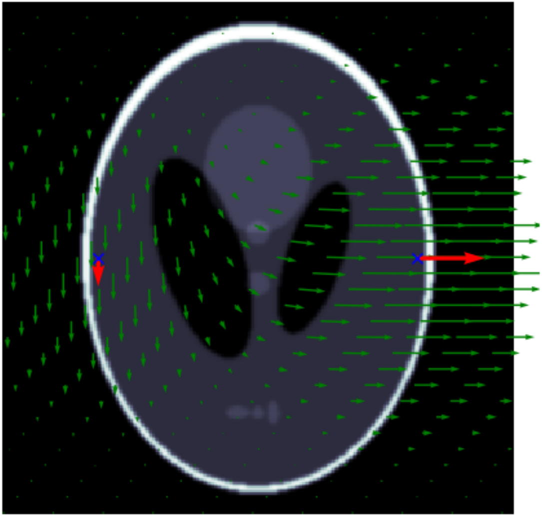

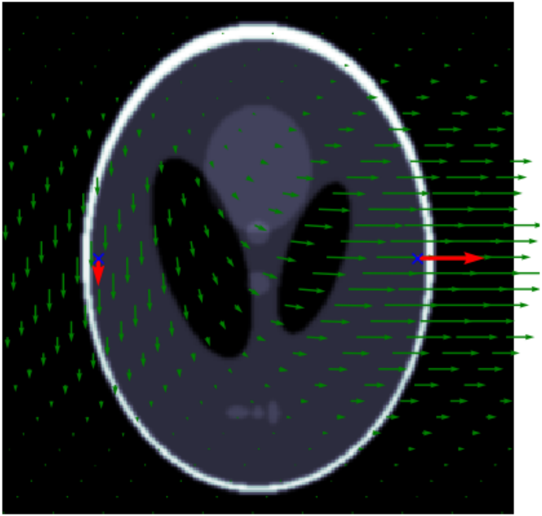

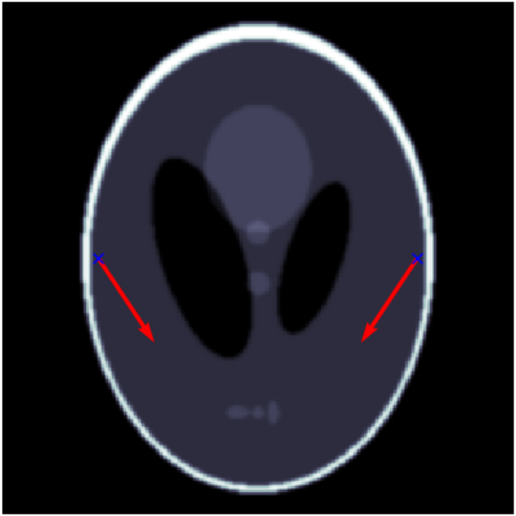

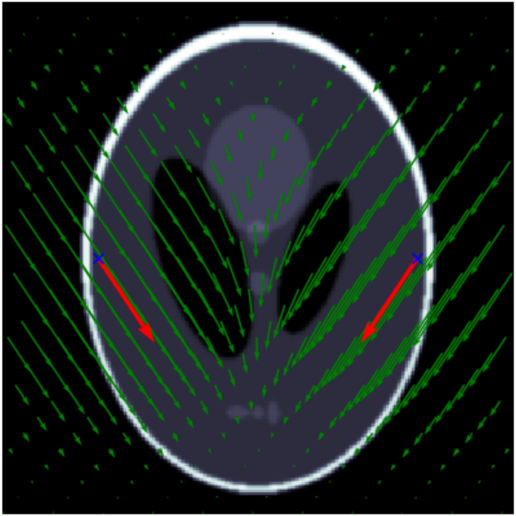

- Incorporate structure in vector fields

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- Easy combination of structures

STRUCTURED LARGE DEFORMATION

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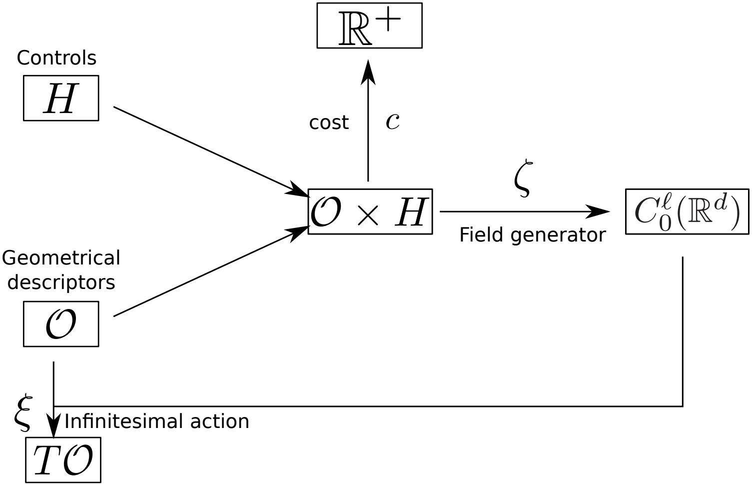
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Definition (Finite energy controlled paths on $\mathcal{O}$)

We denote by Ω the set of mesurable curves $t \mapsto (q_t, h_t) \in \mathcal{O} \times H$ such that :

- $\dot{q}_t = \xi_{q_t}(v_t)$ *where* $v_t = \zeta_{q_t}(h_t) \in \zeta_{q_t}(H)$

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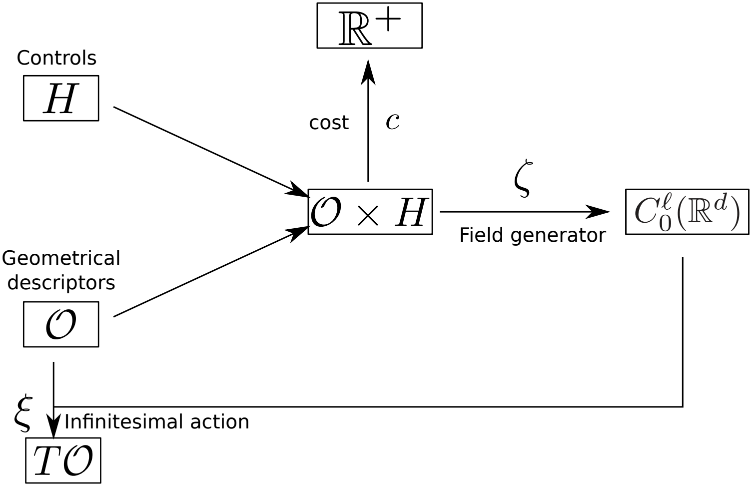
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- *Energy* $E(q, h) \doteq \int_0^1 c_{q_t}(h_t) dt < \infty$

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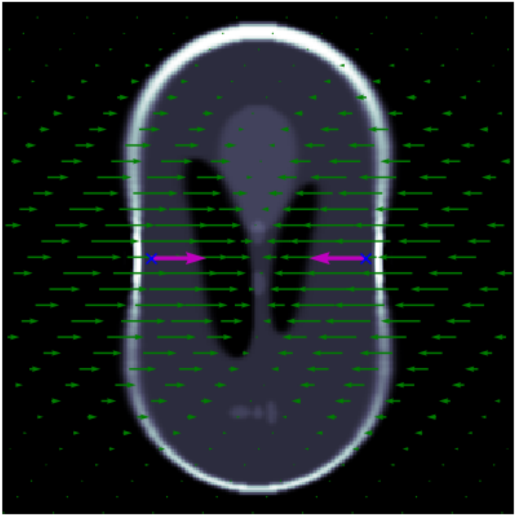
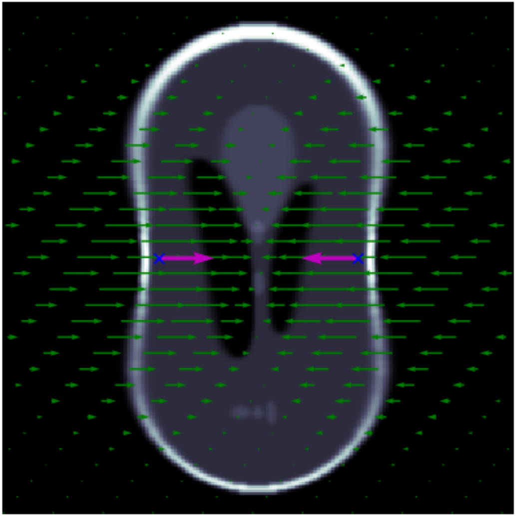
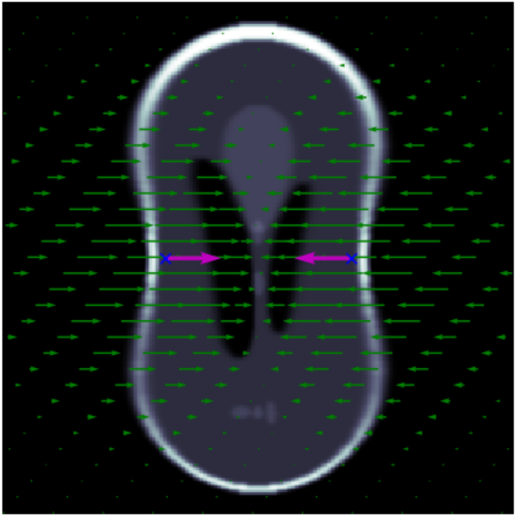
$\longrightarrow \varphi_{t=1}^{\zeta_q(h)}$ is a modular large deformation.

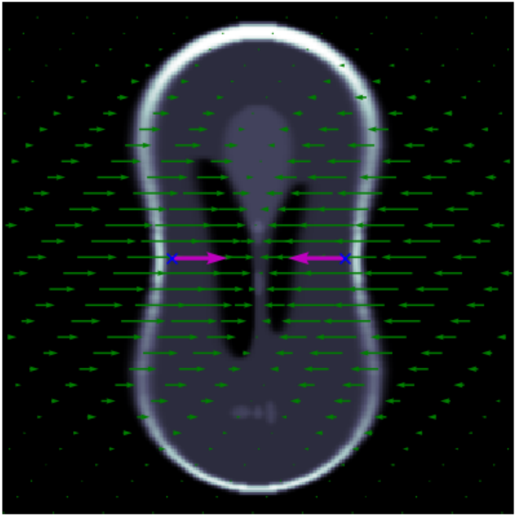
$\longrightarrow \varphi_{t=1}^{\zeta_q(h)} \cdot q_0 = q_1$.

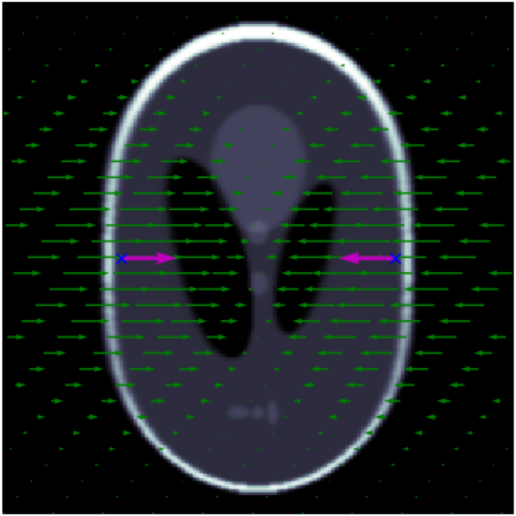
$\longrightarrow \varphi^{\zeta_q(h)}$ is defined by $(q_{t=0}, h) \in \mathcal{O} \times L^2([0,1], H)$.

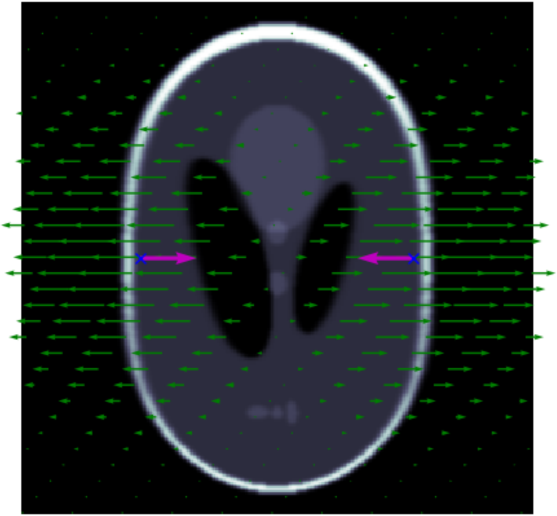


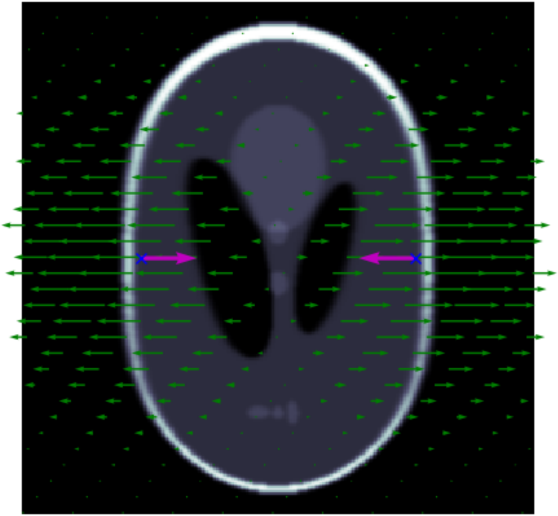


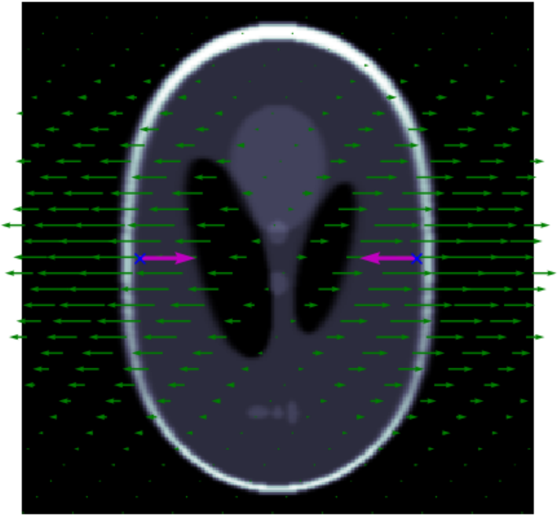


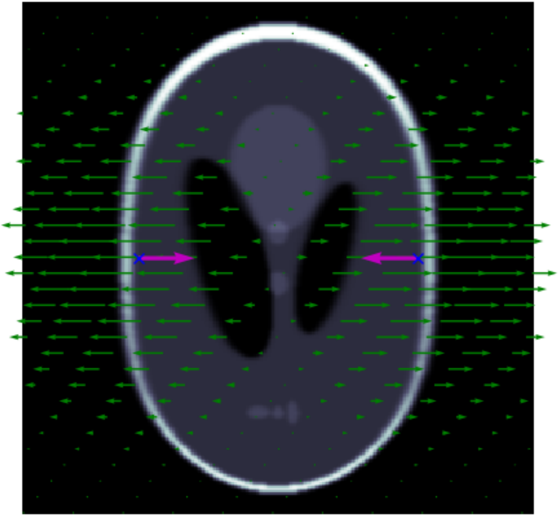


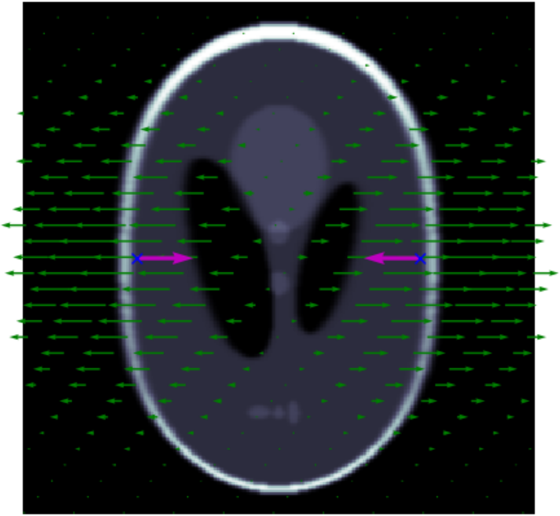


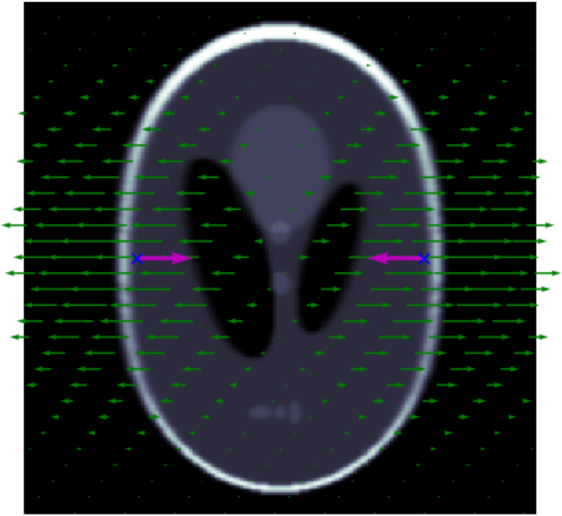


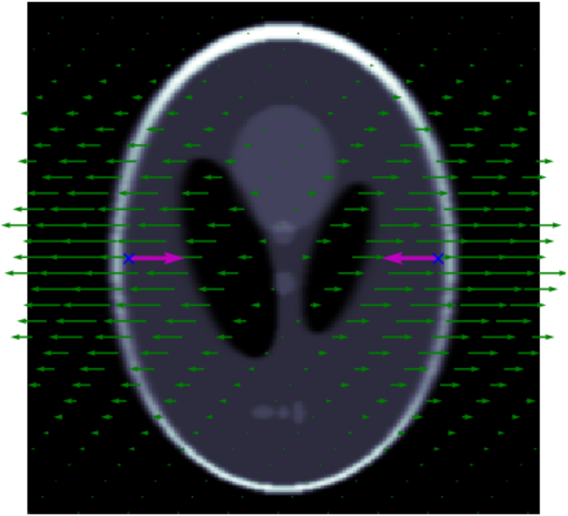


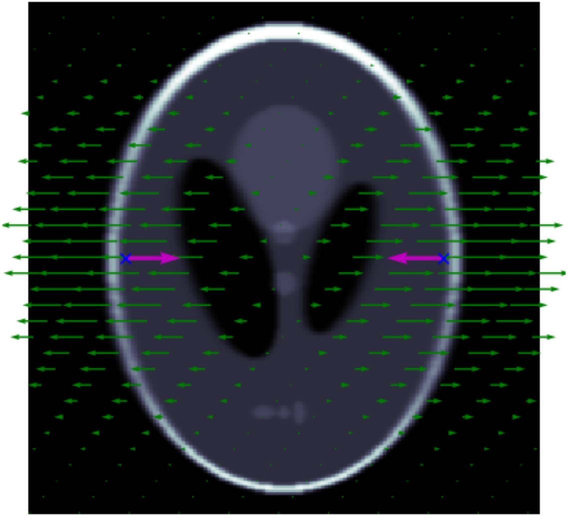


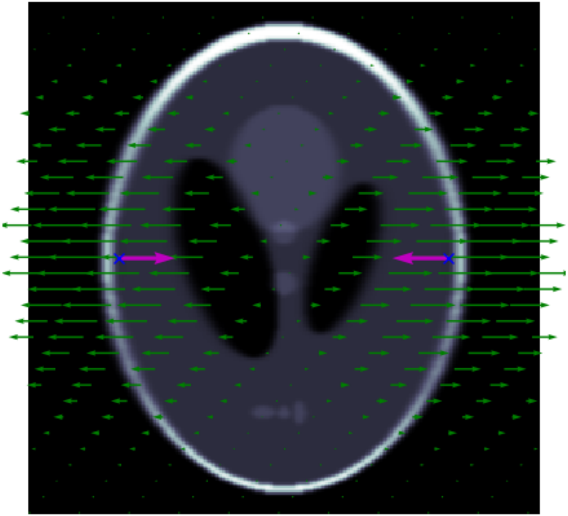












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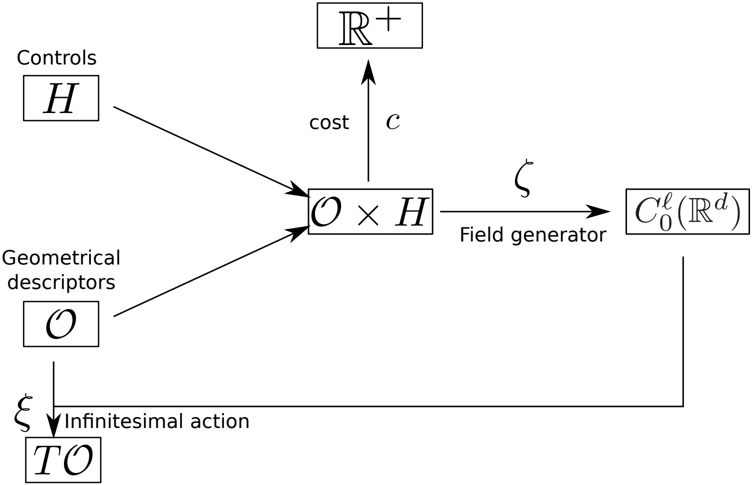
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SUB-RIEMMANNIAN STRUCTURE ON $\mathcal{O}$

Proposition

Wet set $\rho : (q,h) \in \mathcal{O} \times H \mapsto (q, \xi_q \circ \zeta_q(h)) \in T\mathcal{O}$. *Then* $(\mathcal{O} \times H, c, \rho)$ *defines a sub-Riemannian structure on* $\mathcal{O}$ *and*

$$Dist(a,b)^2 = \inf\{\int_0^1 c_q(h) \mid h \in L^2([0,1],H), \dot{q} = \rho_q(h), q_{t=0} = a, q_{t=1} = b\}$$

Theorem

If $Dist(a,b) < \infty$ *the energy* E*, there exists* $(q,h) \in \Omega$ *such that* $q_{t=0} = a$*,* $q_{t=1} = b$ *and* $Dist(a,b) = \sqrt{\int_0^1 c_q(h)}$*.*

[B. G., S. Durrleman, A. Trouvé. A sub-Riemannian modular framework for diffeomorphism based analysis of shape ensembles, SIAM Journal of Imaging Sciences,

10.1137/16M1076733 (2018).]

[S. Arguillère. Géométrie sous-riemannienne en dimension infinie et applications à l'analyse mathématique des formes, PhD thesis, Paris 6, 2014]

[A. Agrachev, D. Barilari, U. Boscain. Introduction to Riemannian and Sub-Riemannian geometry, 2014.]

Proposition

Let $M = (\mathcal{O}, H, \zeta, \xi, c)$ be a deformation modules satisfying the UEC and $\mu : \mathcal{O} \mapsto \mathbb{R}^+$ C^1. Let $a \in \mathcal{O}$ and

$$J_a : h \in L^2([0,1], H) \mapsto \int_0^1 c_{q_t}(h_t)\mathrm{d}t + \mu(q_{t=1}, b)$$

with $q_{t=0} = a$ and (q, h) horizontal.
Minimizers of J_a can be parametrized by an element $\eta \in T_a^*\mathcal{O}$.

APPLICATIONS
IMAGE RECONSTRUCTION

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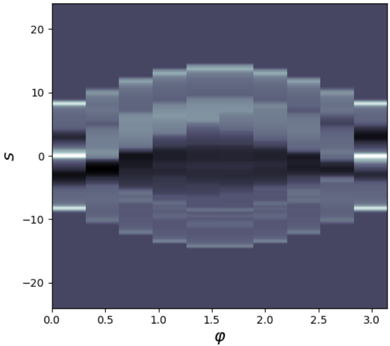
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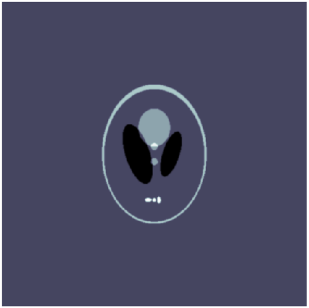

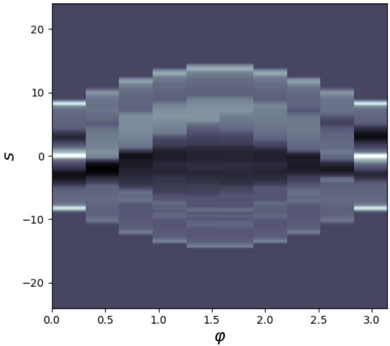
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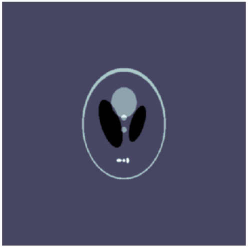
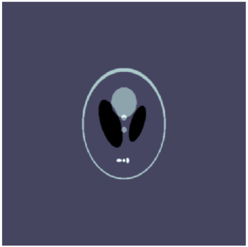
Strategy: using geodesics parametrized by $(a, \eta) \in T_a^*\mathcal{O}$ to transform I_0.

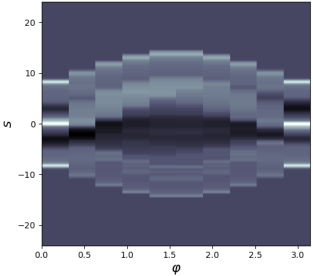
$$\boxed{J_{I_0,g}(a,\eta) = C(a,\eta) + \frac{1}{\lambda} D\Big(T(\varphi_{t=1}^{\zeta_q(h)} \cdot I_0), g \Big)}$$

with (q, h) the geodesic parametrized by (a, η).
$\longrightarrow$ A well-defined regularization method (existence, stability and convergence).

[B.G., Incorporation of a deformation prior in image reconstruction (2018)]



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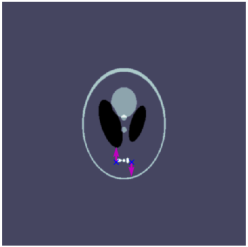

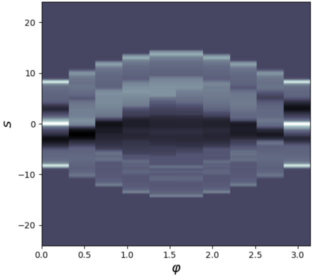
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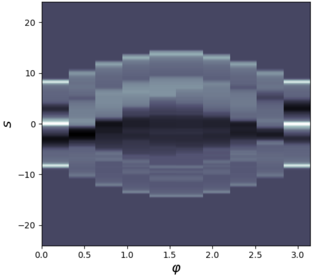
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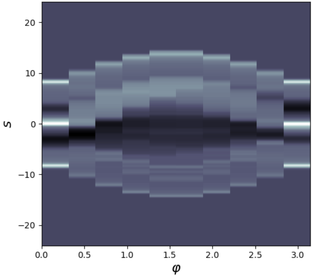
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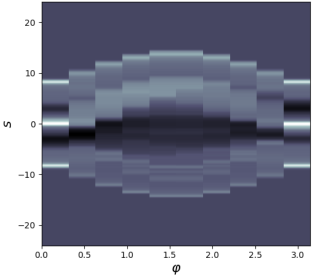
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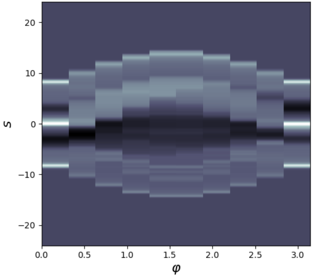
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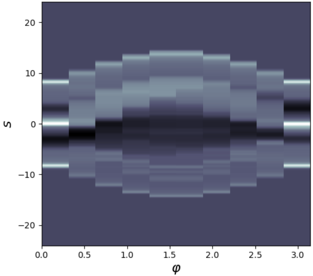
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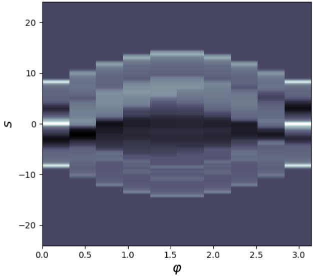
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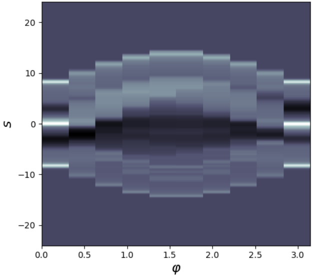
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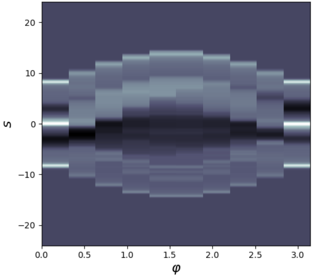
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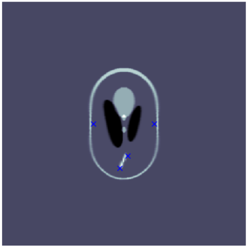

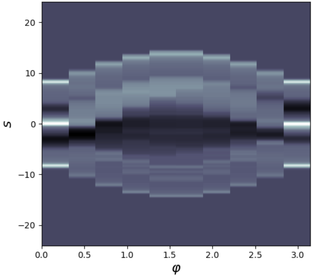
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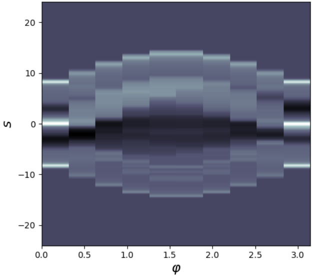
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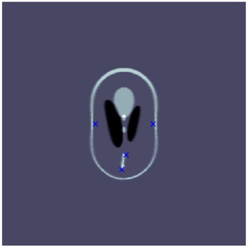

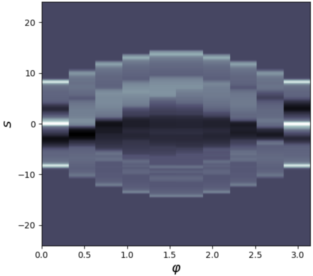
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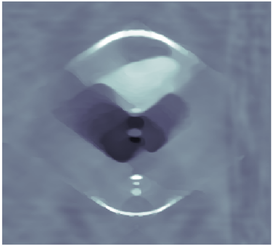

[J. Adler et al. ODL - a Python framework for rapid prototyping in inverse problems. In preparation, KTH, Royal Institute of Technology. Code and documentation available online: https://github.com/odlgroup/odl.

- Incorporate known prior in image reconstruction
- Reconstruction in low dimension

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Questions ?