A "Total Variation" with curvature penalization

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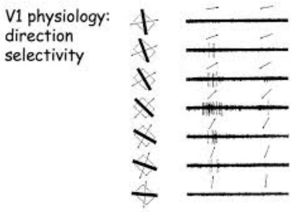
Introduction

- a representation based on the "roto-translation" group;
- a simple formula for curvature-dependent line energies;
- a general relaxation for functions;
- a tightness result (C^2 sets);
- the dual formulation and link with previous works [Bredies-Pock-Wirth'15];
- numerical results

Curvature information: a "natural" idea

Experiments and discovery of Hubel-Wiesel (62, 77)



Observation: the brain reacts to orientation. Corresponding cells are stacked and connected together to provide sensitivity to curvature. First mathematical theories: Koenderink-van Doorn (87), Hoffman (89), Zucker (2000), Petitot-Tondut (98/2003), Citti-Sarti (2003/2006).

Main idea: use the sub-Riemanian structure of the roto-translation group ($(a, R) \in SE(2) \simeq \mathbb{R}^2 \rtimes SO(2) \sim \mathbb{R}^2 \times \mathbb{S}^1$ in dimension 2) to describe the geometry of the visual cortex

$\rightarrow$ sub-Riemanian diffusion and mean curvature motion (Citti-Sarti 3/6, Franken-Duits 09, Boscain et al 14, Citti et al, 2015, ...) for inpainting.

$\rightarrow$ sub-Riemanian length minimization (Mirebeau 2014-17, Boscain et al, 2014, Bekkers et al, 2015, Duits et al, 2014-2016, Chen et al 2017)

(More like this work.)

Variational approaches

Mumford (94) suggested to use the "elastica" functional

$$\int_\gamma \kappa^2 d\mathcal{H}^1$$

for **contour completion**. (Idea suggested by psychological experiments, *cf* for instance Kanizsa 1980.) General theory by Masnou-Morel 98. Issues: not lower semicontinuous. Studied by Bellettini-Mugnai 2004/2005, Nardi (PhD 2011), Dayrens-Masnou 16, Ambrosio-Masnou 2003. *[Examples]*

Variational approaches

• Minimisation of elastica or similar energies is computationally challenging. A few approaches trying to exploit the roto-translation metric: Schoenemann with Cremers (2007), Kahl and Cremers (2009), Masnou and Cremers (2011): discrete approach on a graph (or LP) where vertices encode position and orientation (also, El Zehiry-Grady 2010, ...);

• Length minimization [geodesic curves in RT group] (Mirebeau, 2014, Bekkers et al, 2015, Duits et al, 2014-2016, Chen et al 2017, Mirebeau 2017) (JM Mirebeau: fast marching for solving anisotropic Eikonal equations.)

Remark: representation of such energies with the "Gauss map" is an old theoretical trick (Anzelotti, 1990).

Variational approaches

Functional setting for inpainting/disocclusion: (Masnou-Morel 1998)

$$u \mapsto \int \left(1 + \left|\operatorname{div} \frac{Du}{|Du|}\right|^p\right) |Du|$$

• Bredies-Pock-Wirth 2013, 2015: "vertex" penalization ("TVX") in the functional setting. Then general energies $\int_\gamma f(x, \tau, \kappa)$, f convex, $f \geq 1$.
Need to "lift" the image in $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$ where last component = curvature, with compatibility condition.
This work: a new (and simpler) representation for the latter approach (with $f(\kappa)$).

Example: a C^2 curve

$\gamma(t)$ planar curve, with $|\dot{\gamma}| = 1$ ($\dot{\gamma} = \tau_\gamma$), and $\ddot{\gamma} = \kappa_\gamma \tau_\gamma^\perp$.
Lifted as $\Gamma(t) = (\gamma(t), \theta(t))$ where $\tau_\gamma = (\cos\theta, \sin\theta)$.
Then: the length of $\Gamma(t)$ in $\Omega \times \mathbb{S}^1$ is

- **Finite**: sub-Riemanian structure, local metric is infinite in direction $\theta^\perp$ (we will also take into account orientation);
- Given by $\int_0^L \sqrt{\dot{\gamma}^2 + \dot{\theta}^2} dt = \int_0^L \sqrt{1 + \kappa^2} dt$: encoding curvature penalization information.

Example: a C^2 curve

Let now $f : \mathbb{R} \to \mathbb{R}$ be convex, assume $f \geq 1$, and consider the energy

$$\int_0^L f(\kappa) = \int_0^L f(\dot{\Gamma}^\theta(t))dt.$$

Observe that if one considers a reparametrization $\lambda(s)$, $s \in [0, a]$, of the curve Γ, then $\lambda^x(s)$ is a reparametrization of γ, $\dot{\lambda}^x = |\dot{\lambda}^x|\tau$, $\kappa = d\theta/dt = \dot{\lambda}^\theta ds/dt = \dot{\lambda}^\theta/|\dot{\lambda}^x|$ hence the energy becomes

$$\int_0^L f(\kappa)dt = \int_0^a f(\dot{\lambda}^\theta/|\dot{\lambda}^x|)|\dot{\lambda}^x|ds.$$

Example: a C^2 curve

Denoting σ the measure (charge) in $\mathcal{M}^1(\Omega \times \mathbb{S}^1; \mathbb{R}^3)$ defined by the curve $\Gamma(t)$:

$$\int_{\Omega \times \mathbb{S}^1} \psi \cdot \sigma = \int_0^L \psi(\Gamma(t)) \cdot \dot{\Gamma}(t) dt,$$

one obtains that

$$\int_0^L f(\kappa) = \int_{\Omega \times \mathbb{S}^1} \overline{h}(\sigma^x \cdot \theta, \sigma^\theta)$$

where

$$\overline{h}(s,t) = \begin{cases} sf(t/s) & \text{if } s > 0, \\ f^\infty(t) & \text{if } s = 0, \\ +\infty & \text{else.} \end{cases} \qquad \textit{(Convex)}$$

where $f^\infty(t) = \lim_{s \to 0} sf(t/s)$ is the recession function of f.

Example: a C^2 curve

It is standard that if f is convex lsc, then also h is, with

$$\bar{h}(s,t) = \sup\left\{as + bt : a + f^*(b) \le 0\right\}.$$

In addition, as $\sigma^x = \lambda\theta$ where λ is a positive measure in $\Omega \times \mathbb{S}^1$, introducing for $p = (p^x, p^\theta) \in \mathbb{R}^3$

$$h(\theta, p) = \begin{cases} \bar{h}(p^x \cdot \theta, p^\theta) & \text{if } p^x \cdot \theta = |p^x| \Leftrightarrow p^x \parallel \theta, p^x \cdot \theta \ge 0 \\ +\infty & \text{else,} \end{cases}$$

which encodes the sub-Riemanian structure of $\Omega \times \mathbb{S}^1$: we also have

$$\int_0^L f(\kappa) = \int_{\Omega\times\mathbb{S}^1} \bar{h}(\sigma^x \cdot \theta, \sigma^\theta) = \int_{\Omega\times\mathbb{S}^1} h(\theta, \sigma).$$

Example: a C^2 curve

Now, observe that $-\operatorname{div}\sigma = \delta_{\Gamma(L)} - \delta_{\Gamma(0)}$, in particular if γ is a closed curve or has its endpoints on $\partial\Omega$, then $\operatorname{div}\sigma = 0$.

Obviously, if one considers the marginal $\bar{\sigma} = \int_{\mathbb{S}^1} \sigma^x \in \mathcal{M}^1(\Omega; \mathbb{R}^2)$ defined by

$$\int_{\Omega\times\mathbb{S}^1} (\psi, 0)\cdot\sigma = \int_\Omega \psi\cdot\bar{\sigma}$$

for any $\psi \in C_c(\Omega; \mathbb{R}^2)$, then it also has zero divergence (as it vanishes if $\psi = \nabla\phi$ for some ϕ). In dimension 2, it follows that (assuming Ω is connected) there exists a BV function u such that $Du^\perp = \bar{\sigma}$. In our case, u is the characteristic function of a set E with $\partial E \cap \Omega = \gamma([0, T]) \cap \Omega$.

Generalization to BV functions

One can define for any $u \in BV(\Omega)$

$$F(u) = \inf \left\{ \int_{\Omega \times \mathbb{S}^1} h(\theta, \sigma) \; : \; \operatorname{div} \sigma = 0, \int_{\mathbb{S}^1} \sigma^x = Du^\perp \right\}.$$

If we assume that $f(t) \geq \sqrt{1+t^2}$, then one sees that $\bar{h}(s,t) \geq \sqrt{s^2+t^2}$ and $\int_{\Omega \times \mathbb{S}^1} h(\theta, \sigma) \geq \int_{\Omega \times \mathbb{S}^1} |\sigma|$. It easily follows that the "inf" is a min, and that F defines a convex, lower semicontinuous function on BV with $F(u) \geq |Du|(\Omega)$.
From the example above, we readily see that if E is a C^2 set, then

$$F(\chi_E) \leq \int_{\partial E} f(\kappa) d\mathcal{H}^1.$$

Tightness of the representation

We can show the following result:

Theorem if E is a C^2 set, then

$$F(\chi_E) = \int_{\partial E} f(\kappa) d\mathcal{H}^1.$$

Proof: we need to show $\geq$. In other words, we need to show the obvious fact that if σ is a measure with $\int_{\mathbb{S}^1} \sigma^x = D\chi_E^\perp$, then σ, above ∂E, consists at least in the measure defined by the lifted curve above ∂E (with its orientation as third component).
Maybe there is a simple way to do this (as it is obvious). We used S. Smirnov's theorem which shows that if σ is a measure with $\operatorname{div} \sigma = 0$, then it is a superposition of curves.

Smirnov's Theorem A (1994)

If $\operatorname{div}\sigma = 0$ then it can be decomposed in the following way:

$$\sigma = \int_{\mathfrak{C}_1} \lambda d\mu(\lambda), \quad |\sigma| = \int_{\mathfrak{C}_1} |\lambda| d\mu(\lambda),$$

where λ are of the form

$$\lambda_\gamma = \tau_\gamma \mathcal{H}^1 \llcorner \gamma$$

for rectifiable (possibly closed) curves $\gamma \subset \Omega \times \mathbb{S}^1$ of length at most one.
($\mathfrak{C}_1$ is the corresponding set.)

Smirnov's Theorem A (1994)

Thanks to the fact that the decomposition is convex (ie with $|\sigma| = \int_{\mathfrak{C}_1} |\lambda| d\mu(\lambda)$) we can show that $|\sigma|$-a.e., for μ-a.e. curve λ one has $\sigma/|\sigma| = \lambda/|\lambda|$ $|\lambda|$-a.e., and in particular λ^x is oriented along θ, and

$$\int_{\Omega\times\mathbb{S}^1} h(\theta,\sigma) = \int_{\mathfrak{C}_1}\left(\int_{\Omega\times\mathbb{S}^1} h(\theta,\lambda)\right) d\mu(\lambda) = \int_{\mathfrak{C}_1}\left(\int_{\gamma} h(\theta,\tau_\gamma) d\mathcal{H}^1\right) d\mu(\lambda_\gamma).$$

The horizontal projection λ^x is a rectifiable curve, and one can deduce that its curvature is a bounded measure.
For this we reparametrize λ with the length of λ^x: that is we define $\tilde\lambda(t) = \lambda(s(t))$ in such a way that $\mathcal{H}^1(\tilde\lambda^x([0,t])) = t$) [if simple]. Then we show that $\tilde\lambda^\theta(t)$, which is the orientation of the tangent [because the energy is finite], has bounded variation.

Tightness

Then one can show that if

$$\Gamma^+ = \{x \in \partial E \cap \lambda^x(0, L) : \text{ the curves have the same orientation } \}$$

then a.e. on Γ^+, the absolutely continuous part of the curvature $\kappa = \dot{\tilde{\lambda}}^\theta$ coincides with κ_E. Using that for any set I,

$$\int_{\lambda^x(I)} f(\kappa^a) \leq \int_{I \times \mathbb{S}^1} h(\theta, \lambda),$$

which more or less follows because this is precisely the way we have built h, we can deduce since $\kappa^a = \kappa_E$ a.e.:

$$\int_{\partial E} f(\kappa_E) d\mathcal{H}^1 = \int_{\mathfrak{C}_1} \int_{\partial E \cap \lambda^x} f(\kappa^a) d\mu(\lambda) \leq \int_{\mathfrak{C}_1} \int_{\partial E \times \mathbb{S}^1} h(\theta, \lambda) d\mu(\lambda)$$

which implies our inequality.

Tightness

- More cases?
- We know that F can be below the standard (L^1) relaxation of $\int_{\partial E} f(\kappa)$ (Bellettini-Mugnai 04/05, Dayrens-Masnou 16) (simple examples).

Dual representation

We can compute the dual problem of

$$F(u) = \inf \left\{ \int_{\Omega \times \mathbb{S}^1} h(\theta, \sigma) \,:\, \operatorname{div} \sigma = 0, \int_{\mathbb{S}^1} \sigma^x = Du^\perp \right\}.$$

by the standard perturbation technique, which consists in defining

$$G(p) = \inf \left\{ \int_{\Omega \times \mathbb{S}^1} h(\theta, \sigma + p) \,:\, \operatorname{div} \sigma = 0, \int_{\mathbb{S}^1} \sigma^x = Du^\perp \right\},$$

showing (exactly as for F) that $p \mapsto G(p)$ is (weakly-$*$) lsc and therefore that $G^{**} = G$, and in particular

$$F(u) = G(0) = \sup_{\eta \in C_0^0(\Omega \times \mathbb{S}^1; \mathbb{R}^3)} -G^*(\eta)$$

Dual representation

Then, it remains to compute $G^*(\eta)$:

$$G^*(\eta) = \sup_{\substack{p,\sigma:\operatorname{div}\sigma=0 \\ \int_{\mathbb{S}^1}\sigma = Du^\perp}} \int_{\Omega\times\mathbb{S}^1} \eta\cdot p - h(\theta,\sigma+p)$$

$$= \sup_{\substack{\sigma:\operatorname{div}\sigma=0 \\ \int_{\mathbb{S}^1}\sigma = Du^\perp}} -\int_{\Omega\times\mathbb{S}^1} \eta\cdot\sigma + \sup_p \eta\cdot(\sigma+p) - h(\theta,\sigma+p)$$

We find $\underline{\theta}\cdot\eta^x + f^*(\eta^\theta) \le 0$, and then $\eta = \psi(x) + \nabla\varphi(x,\theta)$ so that:

$$F(u) = \sup\left\{ \int_\Omega \psi\cdot Du^\perp \,:\, \psi\in C_c^0(\Omega;\mathbb{R}^2), \right.$$
$$\left. \exists\,\varphi\in C_c^1(\Omega\times\mathbb{S}^1), \underline{\theta}\cdot(\nabla_x\varphi+\psi) + f^*(\partial_\theta\varphi) \le 0 \right\}.$$

$\rightarrow$ **SAME as Bredies-Pock-Wirth' 2015** (which however is a 5D representation)

Numerical discretization

We tried many different approaches. Best is based on a staggered grid representation (for the image and the gradients) which is 90% justified. But this is probably not the end of the story. Naive or more sound implementations are up to now too diffusive. (The measures σ should be concentrated on lines!)
We use both the primal and dual representation and solve the discretized problem using a saddle-point optimisation.

Examples: shape denoising

Examples: shape denoising

(a) AC, $\lambda = 8$

(b) AC, $\lambda = 4$

(c) AC, $\lambda = 2$

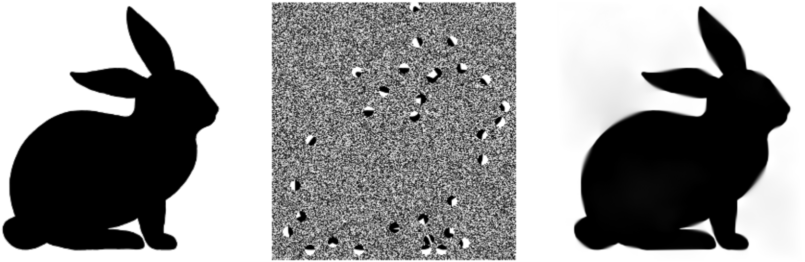
(d) EL, $\lambda = 8$

(e) EL, $\lambda = 4$

(f) EL, $\lambda = 2$

Figure: Shape denoising: First row: Using the function $f_1 = 1 + k|\kappa|$, second row: Using the function $f_3 = 1 + k|\kappa|^2$.

Examples: shape inpainting



(a) Original shape (b) Input (c) Inpainted shape

Figure: (Weickert's) rabbit: Shape completion using the function $f = 1 + |\kappa|^2$.

Examples: shape inpainting

Image with missing parts

(a) Total variation

(b) Elastica

Figure: Inpainting with total variation vs $f = 1 + |\kappa|^2$.

Examples: shape inpainting

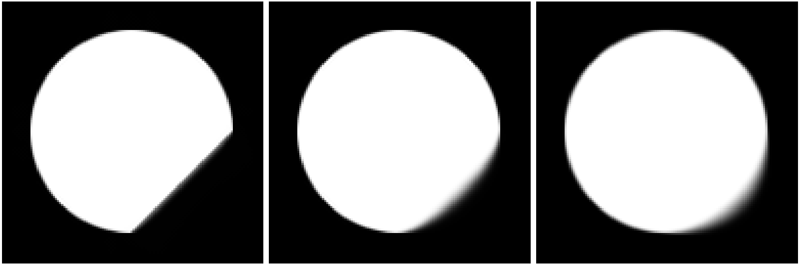
Image with missing lines

(a) Total variation

(b) Elastica

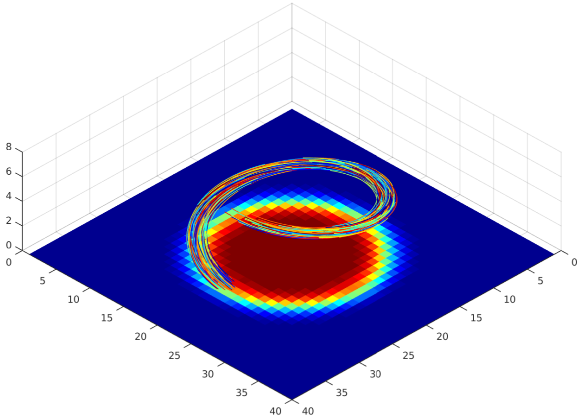
Figure: Inpainting with total variation vs $f = 1 + |\kappa|^2$.

Example: Completion of a disk



Completion of a disk with TV, $1 + |\kappa|^2$, $\varepsilon + |\kappa|^2$

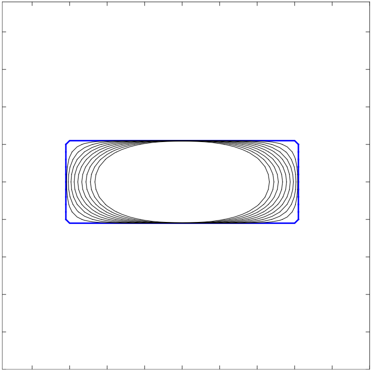
Example: Completion of a disk



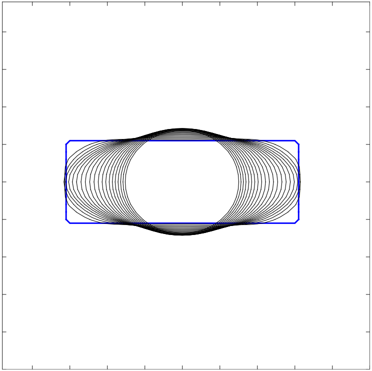
Representation of the disk in the RT space

Examples: Willmore flow

(*cf* for instance Dayrens-Masnou-Novaga 2016)



(a) AC



(b) EL

Figure: Motion by the gradient flow of different curvature depending energies. Energy $1 + |\kappa|$ gives the same as standard mean curvature flow for convex curves. Elastica/Willmore flow converges to a circle (shrinkage is still present due to the length term).

Conclusion, perspectives

- We have introduced a relatively simple systematic way to represent curvature-dependent energies in $2D$;
- It simplifies the (energetically equivalent) framework of [Bredies-Pock-Wirth 15];
- Open questions: characterize the functions for which the relaxation is tight (conjecture: functions with "continuous" curvature?);
- Discretization / Optimization need a lot of improvement.

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Thank you for your attention