# '*Resnikoff's model of perceived colors space*'

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# Color in history

## Earliest philosophical thoughts about nature of color

- **Plato** (-428 $\div$ -348): impossibility of understanding the mechanisms underlying colors now and forever;
- **Aristoteles** (-384 $\div$ -322): necessity of a medium (light) between objects and eyes, colors as a mixture of black and white.

## 17th century scientists and philosophers

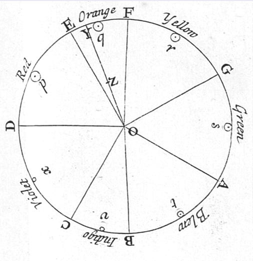
- **Descartes** (1596 $\div$ 1650): colors due to the different rotatory speed of aether particles (fastest $\sim$ Red, slowest $\sim$ Blue);
- **Hooke** (1635 $\div$ 1703): colors originated by the deflection of the light wave front from refractions and reflections of surfaces.

# Color in history

## Newton (1642 ÷ 1726)

- 1671: Decomposition and recomposition of white light by a transparent prism: primitive (no further decomposable) spectral lights;

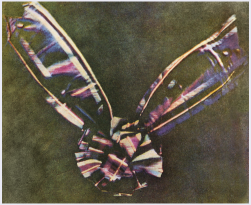
- 1704: *Opticks*, **geometrization of color** (colors as musical notes). Newton's circle: **White**: center; **Saturation**: distance from the center; **Hue** angle (red and violet fuse into purple); **Perimeter**: spectral colors.



# Color in history

## From optics to physiological optics: the trichromatic theory

- **Euler** (1707 ÷ 1783): colors $\longleftrightarrow$ variations of frequency of light waves;
- **Young** (1773 ÷ 1829): hypothesis of nerve fibres in the retina with 3 portions sensitive to a principal color. *Almost* correct: 3 types of *cones* (photoreceptors) responsible to color vision, **Schultze** (1825 ÷ 1874);
- **Helmholtz** (1821 ÷ 1894): rescued Young's intuition from oblivion with experiments that confirmed the **trichromaticity theory** of color vision;
- **Maxwell** (1831 ÷ 1879): light as electromagnetic wave (1861), color photography with RGB filters:



# Color in history

## The modern era of color: Linear algebra and Differential Geometry

- **Grassmann** (1809 $\div$ 1877): abstraction of vector calculus, (1853): the set of perceived colors is a convex cone in a 3-dimensional affine space;
- **Riemann** (1826 $\div$ 1866): in his PhD thesis defense (1854), he quoted the "manifold of perceived colors" as an example of 3-D differential manifold;
- **Open problem**: what kind of Riemannian metric describes color differences?

# Color in history

## Psychophysics and color metrics

- **Weber** (1795 $\div$ 1878) experiments and **Fechner** (1801 $\div$ 1887) formalization: subjective brightness varies as the *logarithm of incident light intensity*;
- **Helmholtz** (1891): proposition of a Riemannian metric coherent with Weber-Fechner's law;
- **Stiles** (1946): generalization of Helmholtz's metric;
- ...longer list: the chemist **Dalton**, the poet **Goethe**, the physiologist **Hering** and the philosophers **Locke**, **Schopenhauer** and **Wittgenstein**.

# Color in history

## Schrödinger's axiomatization

- **Schrödinger** (1887 $\div$ 1961): coherent set of axioms elegantly summarizing previous discoveries (1920);
- Second part of 20th century: prevailed a simpler empirical description of perceived color spaces, CIE (Commission Internationale de l'Éclairage);
- A noticeable exception: (1974)

H.L. **Resnikoff**'s: "***Differential geometry and color perception***"
Journal of Mathematical Biology 1, 97-131.

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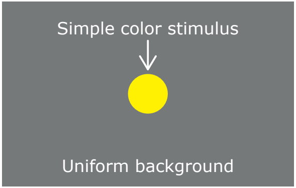
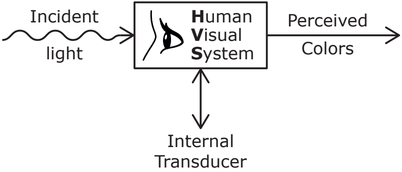
# The setting




- A color stimulus in (the simplest) context;
- The internal (neurophysiological) structure of the HVS is not considered, only its "macroscopical" behavior is;
- The HVS is that of an **ideal average observer** responding to arbitrarily small and large light intensities.

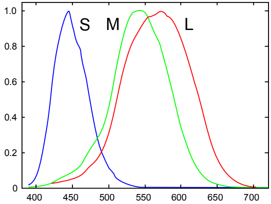
# Metameric classes

## Physical lights

- $\Lambda = [\lambda_{\text{min}}, \lambda_{\text{max}}]$: spectrum of visible wavelenghts;
- $x \in L^2(\Lambda)$: physical light;

## Perceived lights

- $x \sim y$: metameric equivalence;
- $[x]$: **metameric class** of $x$, i.e. physical lights that, integrated w.r.t cones' spectral responses, produce the same tristimulus as $x$.



# The spaces of perceived colors

## The space $\mathcal{P}$

- The quotient space $\boxed{\mathcal{P} = L^2(\Lambda)/\sim}$ is the **space of perceived colors**[a];
- $V$ =span($\mathcal{P}$), with linear structure:

$$\alpha[x] + \beta[y] = [\alpha x + \beta y].$$

- Negative coefficients and equality in $V$ must be interpreted in the framework of color matching.

[a] . . .in the very restrictive conditions specified above.

# Schrödinger's axioms and the basic properties of $\mathcal{P}$

**Axiom 1 (Newton, 1704)**

$x \in \mathcal{P}$, $\alpha \in \mathbb{R}^+ \implies \alpha x \in \mathcal{P}$, i.e. $\mathcal{P}$ is a **cone** in $V$

**Axiom 2 (Schrödinger, 1920)**

$\forall x \in \mathcal{P}$, $\nexists y \in \mathcal{P}$: $x + y = 0$, i.e. no superposition of perceived light is dark (true for natural light, not for the coherent one)

**Axiom 3 (Grassmann, 1853, Helmholtz, 1866)**

$\forall x, y \in \mathcal{P}$, $\forall \alpha \in [0, 1]$, $\alpha x + (1 - \alpha)y \in \mathcal{P}$, i.e. $\mathcal{P}$ is **convex**

**Axiom 4 (Grassmann, 1853)**

$\forall \{x_k,\ k = 1, \ldots, 4\} \subset \mathcal{P}$, $\exists \alpha_k \in \mathbb{R} \setminus \{0\}$: $\sum_{k=1}^{4} \alpha_k x_k = 0$, i.e. $\dim(V) \leq 3$:

3: **trichromate**, 2: **dichromate**, 1: **monochromate**, 0: **blind** observers.

# The fifth axiom: Homogeneity of $\mathcal{P}$

## 1D Motivation

- $X$: topological space, $G$: transformation group, $G \times X \to X$, $(g, x) \mapsto g(x)$, $X$ is a *homogeneous space of G* if, $\forall x, y \in X$, $\exists g \in G$ s.t. $g(x) = y$;
- Weber-Fechner's law: **brightness** $b(x)$ proportional to $\log x$, so relative brightness $b(x_1) - b(x_2)$ of $x_1, x_2$ proportional to

$$\log(x_1) - \log(x_2) = \log(\frac{x_1}{x_2}) = \log(\frac{\lambda x_1}{\lambda x_2}), \quad \forall \lambda \in \mathbb{R}^+.$$

- Relative brightness: invariant under the modification of light intensity

$$x_1 \mapsto \lambda x_1, \quad x_2 \mapsto \lambda x_2, \qquad \lambda > 0.$$

- This allows us to reproduce the sensation of a natural scene on a canvas, TV, movie screen, etc.

# The fifth axiom: Homogeneity of $\mathcal{P}$

## 1D Motivation

- Set of light intensities: $\mathbb{R}^+$, both topological space and group.
- $\mathbb{R}^+$ is a $\mathbb{R}^+$-homogeneous space:

$$\forall x, y \in \mathbb{R}^+, \quad y = \frac{y}{x} x \equiv \lambda x, \quad \lambda \in \mathbb{R}^+.$$

- Relative brightness is a $\mathbb{R}^+$-invariant function defined on $\mathbb{R}^+$.
- Weber-Fechner's law defines the **unique** $\mathbb{R}^+$**-invariant metric on** $\mathbb{R}^+$ (up to re-parameterizations).
- **Goal**: generalize this argument to the entire color space. Color metrics singled out by invariance properties of human vision.

# The fifth axiom: Homogeneity of $\mathcal{P}$

## The group of background transformations

- Any $x \in \mathcal{P}$ can turn into $y \in \mathcal{P}$ *not too different* from $x$ by a change of background, and this process is reversible;
- The group of changes of background illumination:

$$GL_+(\mathcal{P}) = \{g \in GL(V) \; : \; \det(g) > 0, \text{ and } g(x) \in \mathcal{P} \; \forall x \in \mathcal{P}\}.$$

  orientation-preserving invertible endomorphisms of $V$ which preserve $\mathcal{P}$.
- $\mathcal{P}$ is a *locally homogeneous space* of $GL_+(\mathcal{P})$:

$$\forall x \in \mathcal{P} \;\; \exists U(x) \subset \mathcal{P} : \; \forall y \in U(x) \;\; \exists g \in G : \; y = g(x).$$

- Since $\mathcal{P}$ is a convex cone, local is equivalent to global homogeneity.

## Axiom 5

$\mathcal{P}$ is a (globally) **homogeneous space** of $GL_+(\mathcal{P})$

## Consequences on the structure of $\mathcal{P}$

- If $X$ is $G$-homogeneous space w.r.t the action $\eta : G \times X \to X$, and $K$ is the isotropy subgroup[1] in $x$, then the map $\beta : G/K \to X$, $\beta(gK) = \eta(g, x)$ is a diffeomorphism for every fixed $x \in X$.

- In our case, we can write the diffeomorphic identification:

$$\boxed{\mathcal{P} \simeq \mathrm{GL}_+(\mathcal{P})/K}.$$

- $\forall \alpha \in \mathbb{R}^+$, $\alpha \to \alpha x$, preserves $\mathcal{P}$, so $g \in \mathrm{GL}_+(\mathcal{P})$ as $\alpha g'$, $g' \in \mathrm{SL}(\mathcal{P})$.[2]

- So $\mathrm{GL}_+(\mathcal{P}) = \mathbb{R}^+ \times \mathrm{SL}(\mathcal{P})$, and thus

$$\boxed{\mathcal{P} \simeq \mathbb{R}^+ \times \mathrm{SL}(\mathcal{P})/K}.$$

[1] $K = \{g \in G \ : \ g(x) = x\}$, if $X$ is a $G$-homogeneous space, then all isotropy group are conjugated.

[2] $\mathrm{SL}(\mathcal{P})$ is the subgroup of $\mathrm{GL}_+(\mathcal{P})$ given by the matrices of this group with unitary determinant.

## Consequences on the structure of $\mathcal{P}$

- Axiom 4 ($\dim(V) \leq 3$) $\Longrightarrow$ for trichromatic observers $SL(\mathcal{P}) \trianglelefteq SL(3,\mathbb{R})$;
- $3 = \dim(\mathcal{P}) = \dim(\mathbb{R}^+ \times SL(\mathcal{P})/K) = \underbrace{\dim(\mathbb{R}^+)}_{=1} + \underbrace{\dim(SL(\mathcal{P}))}_{\leq \dim(SL(3,\mathbb{R}))=8} - \dim(K)$;
- So: $\boxed{2 + \dim(K) = \dim(SL(\mathcal{P})) \leq 8}$, which allows us determining the possible forms of $SL(\mathcal{P})$ and $K$ (up to isomorphisms).

# Consequences on the structure of $\mathcal{P}$

- Basic idea used by Resnikoff (details in the paper...)

- $\exists$ S, semi-simple Lie group, and $T_{n_i}$, nilpotent Lie groups, $i = 1, \dots, k$, $n_i \in \mathbb{N}$, such that

$$\mathrm{SL}(\mathcal{P}) \simeq S \times (T_{n_1} \times \cdots \times T_{n_k}),$$

where the elements of $T_{n_i}$ are upper triangular matrices of the form

$$T_{n_i} = \left\{ \begin{pmatrix} 1 & & \alpha_{\mu\nu} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} : \alpha_{\mu\nu} \in \mathbb{R}^+, \ 1 \le \mu < \nu \le n_i \right\},$$

whose dimension is $\dim(T_{n_i}) = \frac{n_i(n_i-1)}{2}$, thus

$$\dim(\mathrm{SL}(\mathcal{P})) = \dim(S) + \dim(T_{n_1} \times \cdots \times T_{n_k})$$

and

$$\dim(S) + \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = 2 + \dim(K) \le 8. \qquad (1)$$

# Consequences on the structure of $\mathcal{P}$ : only two possible forms

- $T_{n_i}$ have no compact subgroups, so $K$ is a subgroup of $S$ and verifies the constraint (1).

- The only two semi-simple groups $S$ coherent with this are:

$$\begin{cases} S = \emptyset, \text{ with dimension } 0 \\ S = \mathrm{SL}(2,\mathbb{R}), \text{ with dimension } 3 \end{cases}$$

- If $S = \emptyset$, then $K = \emptyset$ and $T_{n_i}$ are isomorphic to $T_2 = \left\{ \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \, p \in \mathbb{R}^+ \right\}$, $T_2 \simeq \mathbb{R}^+$ hence $\mathcal{P} = \mathrm{GL}_+(\mathcal{P})/K \simeq \mathbb{R}^+ \times \mathrm{SL}(\mathcal{P})/K \simeq \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$;

- If $S = \mathrm{SL}(2,\mathbb{R})$, then $K \simeq \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}, \, 0 \le \vartheta \le 2\pi \right\}$, hence $\mathcal{P} = \mathrm{GL}_+(\mathcal{P})/K \simeq \mathbb{R}^+ \times \mathrm{SL}(\mathcal{P})/K \simeq \mathbb{R}^+ \times \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$.

# Consequences on the structure of $\mathcal{P}$ : only two possible forms

- Summarizing, Axioms 1-5 imply that $\mathcal{P}$ can only have two forms:

1. $\boxed{\mathcal{P} \simeq \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+}$ Helmholtz-Stiles space
    - Perceived colors represented by a triple of positive real numbers (RGB, XYZ, LMS, etc.);

2. $\boxed{\mathcal{P} \simeq \mathbb{R}^+ \times SL(2,\mathbb{R})/SO(2)}$ a new perceptual color space:
    - $\mathbb{R}^+$: *achromatic channel* (average level of intensity);
    - $SL(2,\mathbb{R})/SO(2)$: Poicaré-Lobachevsky 2D space of constant negative curvature, yet to be fully understood in terms of colorimetric attributes.

# Perceptual invariance and color metrics

- It is natural to search for a Riemannian metric on $\mathcal{P}$ coherent with Axioms 1-5: they determine the structure of $\mathcal{P}$ as a homogeneous space;
- $d(x, y)$: perceived difference between lights $x, y$ with a background $b$;
- $d(g(x), g(y))$: perceived difference between lights $g(x), g(y)$ after a change of background from $b$ to $b'$.

**Axiom 6: The perceptual metric**

The Riemannian metric on $\mathcal{P}$ which measures perceptual differences between colors is $\mathrm{GL}_+(\mathcal{P})$-invariant:

$$\boxed{d(g(x), g(y)) = d(x, y)} \quad \forall g \in \mathrm{GL}_+(\mathcal{P}),\ \forall x, y \in \mathcal{P}.$$

## Perceptual metrics on $\mathcal{P}$

- $\mathcal{P} \simeq \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$:

$$\boxed{ds^2 = \alpha_1 \left(\frac{dx_1}{x_1}\right)^2 + \alpha_2 \left(\frac{dx_2}{x_2}\right)^2 + \alpha_3 \left(\frac{dx_3}{x_3}\right)^2}$$

$\alpha_k > 0$, Helmholtz-Stiles's metric.

- $\mathcal{P} \simeq \mathbb{R}^+ \times SL(2,\mathbb{R})/SO(2)$:

$$\boxed{ds^2 = \mathrm{Tr}(x^{-1}dx\, x^{-1}dx)}$$

  - $\mathcal{P} \ni x = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}$ 2×2 positive-definite real symmetric matrix;
  - $x = \det(x) \left(\frac{x}{\det(x)}\right)$, $\det(x) \in \mathbb{R}^+$ and $\frac{x}{\det(x)} \in SL(2,\mathbb{R})/SO(2)$;
  - Tr: *trace* operator, guarantees invariance thanks to its cyclic property.

# Computation of brightness difference with the perceptual metrics

- **Brightness**: perceived difference among lights differing only in intensity: $x$ and $x' = \alpha x$, $\alpha > 0$.

- $\mathcal{P} \simeq \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$:

$$d(x, \alpha x) = \int_x^{\alpha x} ds = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2} \int_1^{\alpha} \frac{dt}{t} = \boxed{\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}\ \log(\alpha)}$$

- If $\mathcal{P} \simeq \mathbb{R}^+ \times SL(2, \mathbb{R})/SO(2)$:

$$d(x, \alpha x) = \int_x^{\alpha x} ds = \sqrt{\mathrm{Tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \int_1^{\alpha} \frac{dt}{t} = \boxed{\sqrt{2}\ \log(\alpha)}$$

- In both cases we recover **Weber-Fechner's law**.

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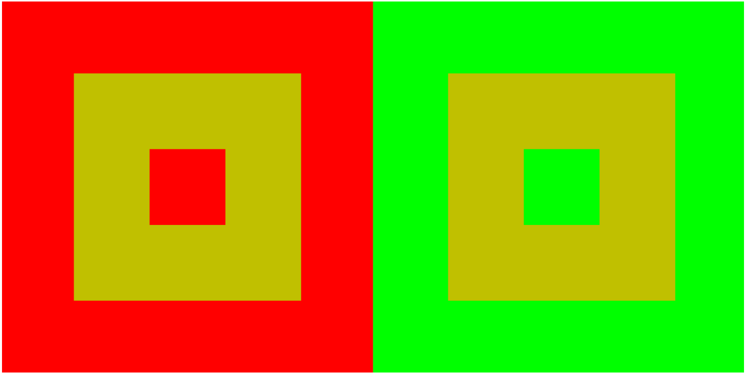
# Two main limits of the Resnikoff model: 1 - Isolated conditions

## Color in context: Induction

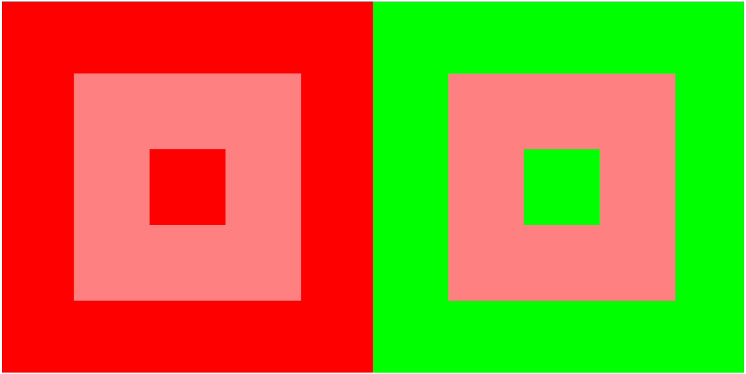
- Real scenes are not given by isolated lights on a uniform background.
- The distribution of illumination and reflectances influences visual perception: **induction**.
- Induction affects all three chromatic attributes:
  - Hue
  - Saturation
  - Brightness

# Induced Hue



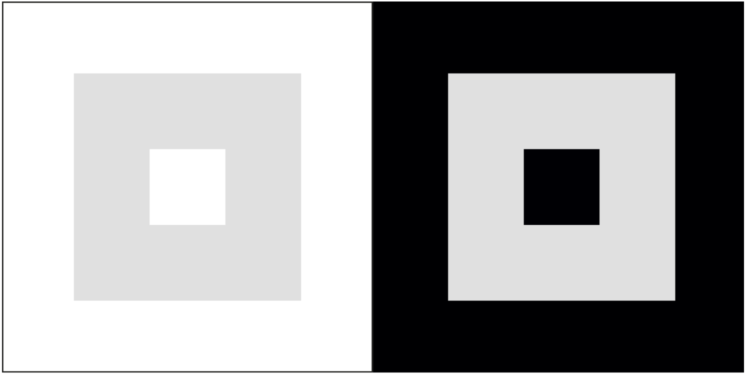
**Induced Hue**

# Induced Saturation



**Induced Saturation**

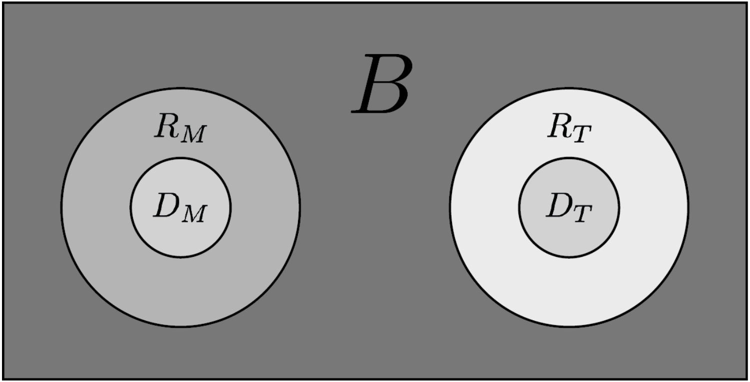
# Induced Brightness (or Achromatic induction)



**Induced Brightness**

# Two main limits of the Resnikoff model: 1 - Isolated conditions

- Induction can be **measured through psychophysical experiments** (Wallach (1948), Rudd-Zemach (2004), Gronchi-Provenzi, (2017)).



# Two main limits of the Resnikoff model: 2 - Group of transformations

## Psychophysical validation of Resnikoff's hypotheses

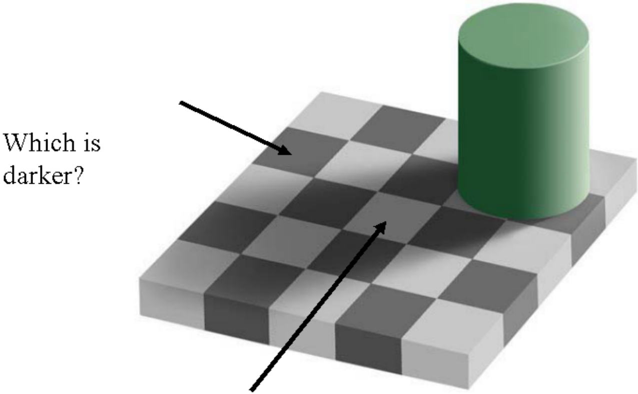
- Resnikoff himself, in a following paper, claimed that

  'the strongest hypothesis about the group of transformations acting on $\mathcal{P}$ is linearity'.

- Up to this date...and my knowledge, no psychophysical experience has been performed to test this yet;

- Taken into account the central role of $GL_+(\mathcal{P})$ in Resnikoff's model, this experiment is crucial.

# But also...(far more difficult) cognitive effects must eventually be taken into account



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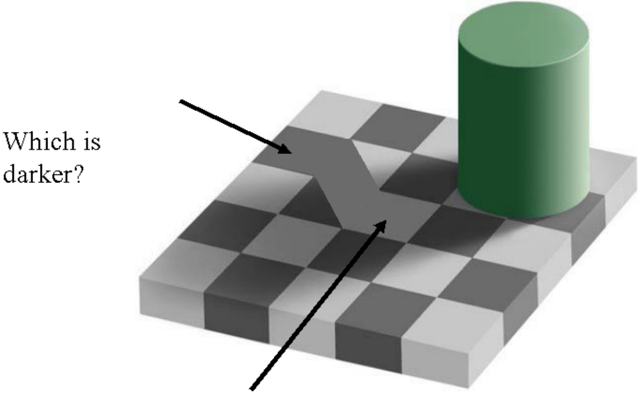

THANKS!